B.Sc. - part-I paper-II
Sub:- Math
Topic:- Integration of irrational function.

Function of form $\sqrt{ax^2+bx+c}$ & $\frac{1}{\sqrt{ax^2+bx+c}}$ are called irrational function

Some standard formula

(I) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}$

(II) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(III) $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}$

(IV) $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\{x+\sqrt{x^2-a^2}\}$

(V) $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2})$

(VI) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log(x+\sqrt{x^2+a^2})$

problem (1) $\int \frac{1+x^2}{1-x^2}\,\frac{dx}{\sqrt{1+x^2+x^4}}$

Soln:- $I = \int \frac{x(\frac{1}{x}+x)}{x(\frac{1}{x}-x)}\,\frac{dx}{\sqrt{x^2(\frac{1}{x^2}+1+x^2)}}$

$= -\int \frac{x+\frac{1}{x}}{x-\frac{1}{x}}\,\frac{dx}{x\sqrt{(x-\frac{1}{x})^2+3}}$

we put $x-\frac{1}{x}=z$ so that

$\left(1+\frac{1}{x^2}\right)dx = dz$

This $\Rightarrow \left(x+\frac{1}{x}\right)\frac{dx}{x} = dz$

$\therefore I = -\int \frac{dz}{x\sqrt{z^2+3}}$ —— (1)

Again we put $z=\frac{1}{y}$ so $dz = -\frac{1}{y^2}dy$

From (1) $I = \int \frac{-\frac{1}{y^2}dy}{\frac{1}{y}\sqrt{\frac{1}{y^2}+3}} = \int \frac{dy}{\sqrt{1+3y^2}}$

$= \int \frac{dy}{\sqrt{3\left(\frac{1}{3}+y^2\right)}} = \frac{1}{\sqrt{3}} \frac{dy}{\sqrt{y^2+\left(\frac{1}{\sqrt{3}}\right)^2}}$

$= \frac{1}{\sqrt{3}} \log\left\{y+\sqrt{y^2-\frac{1}{3}}\right\}$

$= \frac{1}{\sqrt{3}} \sinh^{-1}\left(\frac{y}{\frac{1}{\sqrt{3}}}\right)$

$= \frac{1}{\sqrt{3}} \sinh^{-1}\left(\sqrt{3}\cdot\frac{1}{z}\right) = \frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3}y)$

$= \frac{1}{\sqrt{3}} \sinh^{-1}\left(\frac{\sqrt{3}x}{x^2-1}\right)$

Problem (2) integrate

$\sqrt{(x-\alpha)(\beta-x)}\,dx$

Soln:- Let $I = \int \sqrt{(x-\alpha)(\beta-x)}\,dx$

Here $(x-\alpha)(\beta-x) = x\beta - x^2 - \alpha\beta + \alpha x$

$= -x^2 + x(\alpha+\beta) - \alpha\beta = -\{x^2 - x(\alpha+\beta) + \alpha\beta$

$= -\left[\left\{x^2 - x(\alpha+\beta) + \left(\frac{\alpha+\beta}{2}\right)^2\right\}\right] + \left[\alpha\beta - \left(\frac{\alpha+\beta}{2}\right)^2\right.$

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$$= -\left[\left\{x-\frac{\alpha+\beta}{2}\right\}^2+\left\{\frac{4\alpha\beta-(\alpha+\beta)^2}{4}\right\}\right]$$

$$= -\left[\left\{x-\frac{\alpha+\beta}{2}\right\}^2-\left\{\frac{(\alpha+\beta)^2-4\alpha\beta}{4}\right\}\right]$$

$$= -\left[\left\{x-\frac{\alpha+\beta}{2}\right\}^2-\left\{\frac{\alpha-\beta}{2}\right\}^2\right]$$

$$= \left(\frac{\alpha-\beta}{2}\right)^2-\left(x-\frac{\alpha+\beta}{2}\right)^2$$

Let $\frac{\alpha-\beta}{2}=a$, $x-\frac{\alpha+\beta}{2}=y$ then $dx=dy$

$$(x-\alpha)(\beta-x)=a^2-y^2$$

Now $I=\int\sqrt{(x-\alpha)(\beta-x)}\,dx=\int\sqrt{a^2-y^2}\,dy$

$$=\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}$$

$$=\frac{x-\frac{\alpha+\beta}{2}}{2}\sqrt{(x-\alpha)(\beta-x)}+\frac{1}{2}\left(\frac{\alpha-\beta}{2}\right)^2\sin^{-1}\frac{x-\frac{\alpha+\beta}{2}}{\frac{\alpha-\beta}{2}}$$

$$=\frac{2x-\alpha-\beta}{4}\sqrt{(x-\alpha)(\beta-x)}+\frac{(\alpha-\beta)^2}{8}\sin^{-1}\frac{2x-\alpha-\beta}{\alpha-\beta}$$

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